strongly from one at the ends of the contact zone. Graphs of its change at the ends of the contact zone are given in Fig. 3 as a function of the magnitude of the zone and the

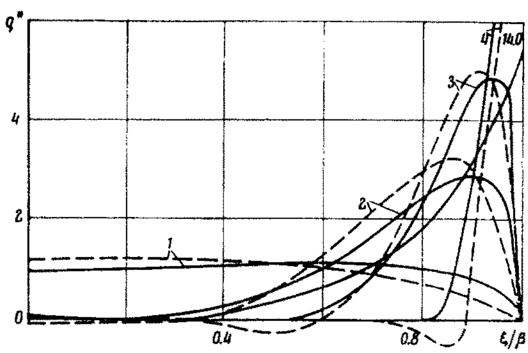

Fig. 2

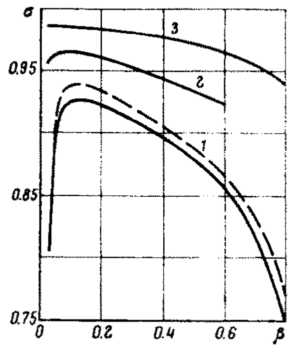

Fig. 3

thickness of the plate. The values of $2l / h = 20, 60, 100$ correspond to curves *1—3* . Shown by dashes is the solution by Kirchhoff theory, but taking account of the nature of the change in reaction in the contact zone, obtained from (2. 4). As we see, the true stresses in the contact zone differ insignificantly for thin plates from those obtained by Kirchhoff theory.

REFERENCES

1. Keer, L. M. and Silva, M. A. G. , Bending of a cantilever brought gradually into contact with a cylindrical supporting surface. Intern, J, Mech. Sci. , Vol. 12, pp. 751-760, 1970.
2. Filonenko-Borovich, M. M. , Theory of Elasticity. 4-th Ed. , Fizmatgiz, Moscow, 1959.
3. Tolkachev, V. M. , Action of sharp dies on a cylindrical shell of infinite length. PMM Vol. 35, №4, 1971.
4. Krylov, V. I. , Handbook on Numerical Integration. "Nauka", Moscow, 1966.
5. Filonenko-Borovich, M. M. , Bending of a thin bar along a given curve. Trudy Mosk. Electro-mekhan. Inst. Inzh. Zh. -D. Transp. , № 58, 1949.

Translated by M. D. F.

UDC 532. 3; 534. 1

WAVE EXCITATION IN A LAYER BY A VIBRATING STAMP

PMM Vol. 39, № 5, 1975, pp. 884-888
V. A. BABESHKO and V. E. VEKSLER
(Rostov-on-Don)
(Received February 5, 1975)

The problem of the vibrations of a rigid circular stamp on the surface of an elastic layer at rest on a rigid base is examined. There is no friction between the stamp and the layer, and between the layer and the base. The contact stresses

under the stamp and the elastic waves originating outside the stamp are studied. A method is proposed for solving these problems at all fundamental frequencies with the exception of some singular frequencies for which another approach is necessary.

The method used is based on the reduction of boundary value problems to an integral equation of the first kind, which differs from the equations investigated in static problems by strong oscillation of the kernel as well as by its bounded growth at certain frequencies. This makes known methods of investigating integral equations ineffective. The method proposed here, based on a special factorization of functions, permits overcoming the mentioned difficulties.

1. Use of the principle of limit absorption [1] reduces the problem to the solution of an integral equation of the following kind:

$$\int_0^a k(r, \rho)\, \rho q(\rho)\, d\rho = \theta u(r), \quad 0 \leqslant r \leqslant a \tag{1.2}$$

$$k(r, \rho) = \int_\sigma uK(u, \varkappa_2) J_s(ur) J_s(u\rho)\, du, \quad s = 0, 1, 2, \ldots \tag{1.3}$$

$$K(u, \varkappa_2) = [u^2\theta_2 \operatorname{cth}\theta_2 - (u^2 - {}^1/_2\varkappa_2{}^2)^2\, \theta_1{}^{-1} \operatorname{cth}\theta_1]^{-1}$$

$$\theta_k = \sqrt{u^2 - \varkappa_k{}^2}, \quad \varkappa_1{}^2 = \rho\omega^2 h^2 (\lambda + 2\mu)^{-1}, \quad \varkappa_2{}^2 = \rho\omega^2 h^2 \mu^{-1}$$

$$\theta = 4\ \mu h b^2 \omega^{-2}, \quad a = R / h, \quad b^2 = \mu / \rho$$

Here $\operatorname{Re} q(r)e^{-i\omega t}$ and $\operatorname{Re} u(r)e^{-i\omega t}$ are contact stresses under the stamp and the characteristic of its shape and the character of insertion, respectively; ω, ν, λ, μ are the stamp vibration frequency, Poisson's ratio of the layer material and Lamé constants, respectively; h is the thickness of the elastic layer, and R is the radius of the stamp.

A detailed study of the properties of the function $K(u, \varkappa_2)$ preceded the derivation of (1. 1). Namely, the neutral curves of the functions K and K^{-1}, whose graphs for $\nu = 0.2$ are presented in Fig. 1, were studied. The solid lines in Fig. 1 show the neutral curves (i. e. the zeros) of the function $K^{-1}(u, \varkappa_2)$, while the dashes pertain to $K(u, \varkappa_2)$.

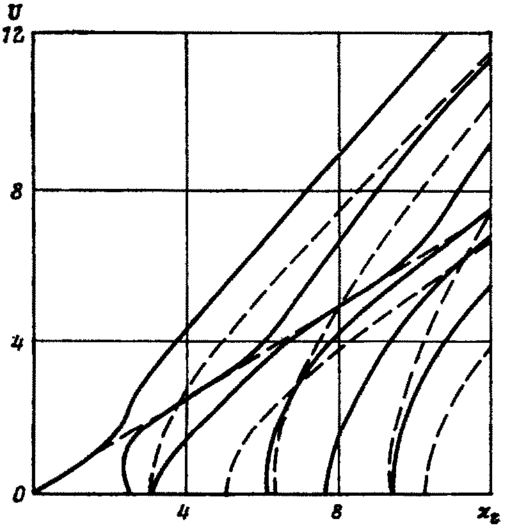

Fig. 1

The principle of limit absorption dictates the location of the contour σ which should agree with the positive part of the real axis everywhere, except at segments containing real poles. In the case zeros and poles of the function $K(u, \varkappa_2)$ alternate, the mentioned segments are bypassed by the contour from below. In Fig. 1 these are sections of values of the parameter $\varkappa_2$ for which the angle between the tangent to the solid curves and the $\varkappa_2$-axis is acute.

The segments containing poles are bypassed from above if the mentioned angle is obtuse. This refers to the least pole which is in the interval $2 < \varkappa_2 < 3$ in Fig. 1. Finally, in the case of multiple poles, the contour should intersect them. Such a position of the contour of integrations is established without special difficulty by using the principle of limit absorption and taking account of the analyticity of u as a function of the parameter $\varkappa_2$. Namely, by imposing a small internal friction ε on the system and using the Cauchy-Riemann relationship on the functions $u(\varkappa_2 + i\varepsilon) = u_1 + iu_2$, we easily obtain that the poles bypassed by the contour from below are shifted to the upper half-plane and those bypassed from above shift to the lower half-plane.

The function $K(u, \varkappa_2)$ decreases as Cu^{-1} at infinity.

To solve (1. 1), let us represent its right side by a Bessel integral, which permits us to be limited to the case when the function $J_0(\eta, r)$, $\eta \geqslant 0$ is in the right side. The solution of the integral equation (1. 1) can be represented as follows for this right side:

$$q_\eta(r) = J_s(\eta r) / K(\eta) + S(r)\varphi \tag{1. 3}$$

Here φ is the solution of an equation of the form

$$\varphi + F\varphi = D \tag{1. 4}$$

The notation used has been introduced in [2]

$$S(r)\varphi = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{I_s(itr)\varphi(t)\,dt}{I_s(ita)K_+(t)} \tag{1. 5}$$

$$F(a, z)\varphi = \frac{1}{(2\pi i)^2}\int\limits_{\Gamma_2}\int\limits_{\Gamma_1} \frac{P(t_2, t_1)\varphi(t_1)\,dt_1\,dt_2}{z - t_2}, \quad z \in \Gamma_3$$

$$D(t) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{K_+(\tau)\Psi(\tau)}{t - \tau}\,d\tau, \quad t \in \Gamma_2 > \Gamma, \ \Gamma_3 > \Gamma_2 > \Gamma_1$$

$$\Psi(\tau) = \left[\frac{\eta K_{s+1}(i\tau a) J_s(\eta a)}{K_s(i\tau a)} + i\eta J_{s+1}(\eta a)\right]\frac{1}{(\eta^2 - \tau^2)K(\eta)}$$

$$P(t_2, t_1) = K_+(t_2)[R_1(t_1) + R_2(t_2)] / (t_2^2 - t_1^2)K_+(t_1)$$
$$R_1(t) = tI_{s+1}(\mathrm{ita})I_s^{-1}(\mathrm{ita}) - t, \qquad R_2(t) = tK_{s+1}(\mathrm{ita})K_s^{-1}(\mathrm{ita}) - t$$

The result of [2] is obtained as a particular case from the representation (1. 3)—(1. 5) if (1. 4) is solved by successive approximations. The validity of this representation can be confirmed by direct substitution into (1. 1).

The main difficulty in effective utilization of this representation is related to the need for factorization of the function $K(u, \varkappa_2)$ relative to the contour σ [3]. To overcome this difficulty, let us perform an approximate factorization by using the approximating function $H(u)$ selected from the following condition:

$$|K(u, \varkappa_2) - H(u)| / |K(u, \varkappa_2)| < \varepsilon$$

In this case, it can be shown exactly as in [4] that for sufficiently small ε the closeness of the solutions will hold in some uniform metric. The approximation is carried out according to the following scheme: let x_i and p_j be positive zeros and poles, respectively, while z_s are complex zeros of the function

$$K\,(u,\,\varkappa_2),\ i=1,\,2,\,\ldots\,n,\ j=1,\,2,\,\ldots\,p.$$

Let us form the function

$$R\,(u)=K\,(u,\,\varkappa_2)\sqrt{u^2+B^2}\prod_{j=1}^{p}(u^2-p_j^2)\prod_{i=1}^{n}\frac{1}{u^2-x_s^{\,2}}\times\begin{cases}\displaystyle\prod_{s=1}^{p-n}\frac{1}{u^2-z_s^{\,2}}, & p>n\\ 1, & p=n\end{cases}$$

which has no real zeros and poles.

The selection of B will be mentioned later; $R\,(u)\sim C$ as $u\to\infty$: $C\neq 0$. For uniqueness of the function $R\,(u)$ let us draw a slit from Bi to $+\,i\infty$ and from $-Bi$ to $-i\infty$ and let us fix the branch by the condition $\sqrt{B^2}=B>0$. Let us make the substitution $x=u^2\,/\,(u^2+M^2)$ which transfers $(0,\,\infty)$ to (0.1) and let us approximate the function $R\,(M\sqrt{x\,/\,(1-x)})$ by the Bernshtein polynomials $B_s(x)$ on the segment (0.1) (the selection of the parameter M influences the accuracy of the approximation substantially). We obtain

$$H\,(u)=B_k\Bigl(\frac{u^2}{u^2+M^2}\Bigr)\frac{1}{\sqrt{u^2+B^2}}\prod_{i=1}^{n}(u^2-x_i^2)\prod_{j=1}^{p}\frac{1}{u^2-p_j^2}\times$$

$$\begin{cases}\displaystyle\prod_{s=1}^{p-n}(u^2-z_s^2), & p>n\\ 1, & p=n\end{cases}$$

The function $H\,(u)$ has $2k$ complex zeros determined by the Bernshtein polynomial, which are symmetric relative to all axes, $2n$ simple real zeros $\pm\,x_m$ and $p-n$ simple complex zeros for $p>n$. The poles of the function $H\,(u)$ are exhausted by two k-tuples $\pm\,Mi$ and $2p$ real $\pm\,p_s$. Moreover, among the singularities of the function $H\,(u)$ are two branch points $\pm\,iB$.

Henceforth, the real zeros and poles are numbered first in order of growth of the moduli, and then all the rest are numbered.

2. By constructing the approximating function we can construct an approximate solution of the problem. To this end, let us drop the contours $\Gamma_1,\,\Gamma_2,\,\Gamma_3$ in (1.4),(1.5) down to the branch point $-\,iB$. The zeros and poles of the integrand will hence intersect. According to residue theory, the functions $F\,(a,\,z)\varphi$ and $D\,(z)$ change form and can be represented as

$$F\,(a,\,z)\,\varphi=B_n+\varepsilon_n,\quad D\,(t)=D_p\,(t)+\varepsilon_p{}',\quad B_n=\sum_{s=1}^{n}f_s(z)\,\varphi(-\,z_s)\qquad(2.1)$$

$$f_s(z)=\sum_{j=1}^{p}\frac{R_1\,(-\,z_s)+R_2\,(-\,p_j)}{(p_j^2-z_s^2)\,K_+{}'\,(-\,z_s)}\Bigl[\Bigl(\frac{1}{K_+\,(-\,p_j)}\Bigr)'\Bigr]^{-1}\frac{1}{z+p_j}-$$

$$\frac{R_1\,(-\,z_s)+R_2\,(-\,Mi)}{(k-1)!\,(M^2+z_s^2)\,K_+{}'\,(-\,z_s)}\,\frac{1}{z+Mi}\lim_{u\to-Mi}\frac{d^{k-1}}{du^{k-1}}\,[K_+\,(u)\,(u+M_i)^k]$$

$$\varepsilon_n=\frac{1}{(2\pi i)^2}\int\limits_{\Gamma_2{}^k}\int\limits_{\Gamma_1{}^k}\frac{K_+\,(t_2)\,[R_1\,(t_1)+R_2\,(t_2)]\,\varphi\,(t_1)\,dt_1\,dt_2}{(t_2^2-t_1^2)\,K_+\,(t_1)\,(z-t_2)}+$$

$$\frac{1}{2\pi i}\sum_{j=1}^{p+1}\frac{1}{z+p_j}\mathop{\mathrm{Res}}_{u=-p_j}K_+\,(u)\int\limits_{\Gamma_1{}^k}\frac{R_1\,(t_1)+R_2\,(-\,p_j)}{(p_j^2-t_1^2)\,K_+\,(t_1)}\,\varphi\,(t_1)\,dt_1+$$

$$\frac{1}{2\pi i}\sum_{i=1}^{n}\frac{\varphi(-z_i)}{K_+'(-z_i)}\int_{\Gamma_2^k}\frac{R_1(-z_i)+R_2(t_2)}{(z-t_2)(t_2^2-z_i^2)}K_+(t_2)\,dt_2$$

$$D_p(t)=\sum_{j=1}^{p+1}\frac{\Psi(-p_j)}{t+p_j}\operatorname*{Res}_{u=-p_j}K_+(u),\quad \varepsilon_p'=\frac{1}{2\pi i}\int_{\Gamma^p}\frac{K_+(\tau)\Psi(\tau)}{t-\tau}\,d\tau$$

For sufficiently large B the influence of the integral terms in the last relationships is slight and they can be neglected. Consequently, to determine the function $\varphi(t)$ it is sufficient to know its value at the points $-z_k$, $K(z_k)=0$. These values are determined from the solution of a finite linear system of algebraic equations of the following kind:

$$\sum_{s=1}^{n}[f_s(-z_i)+\delta_{is}]\,\varphi(-z_s)=D_s(-z_i),\quad i=1,2,\dots n \tag{2.2}$$

For sufficiently large n, which will hold for large B, this system turns out to be solvable uniquely. Having determined $\varphi(-z_s)$, we find an approximate representation of the solution in the inner contact domain and the wave field sufficiently far from the stamp boundary in the form

$$q_\eta(\rho)=\theta\left[\frac{J_s(\eta\rho)}{K(\eta)}+\sum_{j=1}^{n}\frac{I_s(-iz_j\rho)\,\varphi(-z_j)}{I_s(-iz_ja)\,K_+'(-z_j)}\right] \tag{2.3}$$

$$\sigma_z(\rho,t)=\operatorname{Re}e^{-i\omega t}\int_0^\infty \eta G(\eta)\,q_\eta(\rho)\,d\eta \tag{2.4}$$

$$u(r)=\int_0^\infty \eta G(\eta)J_0(\eta r)\,d\eta,\quad 0\leqslant r\leqslant a$$

$$u(r,t)=\operatorname{Re}e^{-i\omega t}\left[\sum_{k=1}^{p}A_kH_s^{(1)}(rp_k)+\sum_{k=p+1}^{n}A_kK_s(-irp_k)\right],\quad r>a \tag{2.5}$$

$$A_k=i\zeta_ka\left[\int_0^\infty \eta\frac{G(\eta)}{K(\eta)}\frac{\eta I_{s+1}(-i\eta a)I_s(-ip_ka)-p_kI_s(-i\eta a)I_{s+1}(-ip_ka)}{\eta^2-p_k^2}\,d\eta+\right.$$

$$\left.\int_0^\infty \eta G(\eta)\sum_{m=1}^{n}\varphi(-z_m)\frac{z_mI_{s+1}(-iz_ma)I_s(-ip_ka)-p_kI_s(-iz_ma)I_{s+1}(-ip_ka)}{I_s(-iz_ma)K_+'(-z_m)(z_m^2-p_k^2)}\,d\eta\right]$$

$$\zeta_k=\operatorname*{Res}_{u=-p_k}uK(u)$$

The first sum is the wave field in the far zone while the second (which damps exponentially as $r\to\infty$) corrects it during approach to the stamp.

As an illustration, let us consider the case of a plane stamp. The equation for the problem of a plane stamp is a particular case of (1.1) with right side in the form of the Bessel function $\alpha J_0(\eta r)$ for $\eta=0$, where α is the depth of insertion of the stamp

$$q(\rho)=\alpha\theta\left[B_0+\sum_{k=1}^{n}B_kI_0(-iz_k\rho)\right] \tag{2.6}$$

$$B_0=K^{-1}(0),\quad B_k=\frac{\varphi(-z_k)}{I_0(-iz_ka)K_+'(-z_k)}$$

The wave field outside the stamp is representable as

$$u(\rho, t) = \alpha \operatorname{Re} e^{-i\omega t} \sum_{j=1}^{p} A_j H_0^{(1)}(p_j \rho) \tag{2.7}$$

Let us consider the case $\nu = 0.2$, $a = 1$, $x_2^2 = 11$. In this case the function $K(u)$ has the poles $p_1 = 0.4984$, $p_2 = 2.004$, $p_3 = 3.393$ and the zeros $z_1 = 2.031$, $z_2 = 1.063$ on the real semi-axis. For the approximation $M^2 = 110$, $B = 15$, $z_3 = 2.397\, i$. The error will not exceed 6% for approximation by a fourth degree polynomial.

Formulas (2.6) and (2.7) permit computation of the stresses under the stamp in a domain not adjoining its edge. Upon approaching the edge of the stamp, the stresses grow as $(r - a)^{-1/2}$ and this singularity can easily be isolated by using the method [5], for example.

Waves on the layer surface in the far zone are computed by means of (2.7). To compute the field in the nearest zone it is necessary to use (2.5); this formula can describe the wave field in a zone arbitrarily near to the stamp because the parameter B increases.

REFERENCES

1. Tikhonov, A. N. and Samarskii, A. G., Equations of Mathematical Physics. (English translation), Pergamon Press, Book № 10226, 1963.
2. Babeshko, V. A., Asymptotic properties of the solution of a class of integral equations of elasticity theory and mathematical physics. Dokl. Akad. Nauk SSSR, Vol. 193, № 3, 1970.
3. Babeshko, V. A., On the theory of dynamical contact problems. Dokl. Akad. Nauk SSSR, Vol. 201, № 3, 1971.
4. Babeshko, V. A., Integral equations of convolutions of the first kind on a system of segments occurring in the theory of elasticity and mathematical physics. PMM Vol. 35, № 1, 1971.
5. Babeshko, V. A. and Garagulia, V. A., Asymptotic solution of the problem of the effect of a circular stamp on an elastic layer. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, № 1, 1971.

Translated by M. D. F.

UDC 539.3: 534.1

OPTIMIZATION OF VIBRATION FREQUENCIES OF AN ELASTIC PLATE IN AN IDEAL FLUID

PMM Vol. 39, № 5, 1975, pp. 889-899

N. V. BANICHUK and A. A. MIRONOV
(Moscow)

(Received October 31, 1974)

The problem of optimizing the frequencies of an elastic plate vibrating in an ideal fluid is investigated. A formulation of the appropriate hydroelasticity problem is presented. The "external" hydrodynamic problem is solved by methods of complex variable function theory and the forces exerted by the fluid on the plate are determined. An integro-differential equation describing one-dimensi-